Lesson 4.3.1

# The Natural Exponential and Logarithm

SACWAY
BRAINSTORMING ALGEBRA & STATISTICS

STUDENT NAME ____________________ DATE ____________

## INTRODUCTION

### Review

In our previous lesson, we learned that fixed-rate interest earning accounts grow according to the compound interest formula,

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this formula, A is account balance at time t, *P* is the principal (the initial investment), r is the annual interest rate, and *n* is the number of times that the account earns interest each year.

## TRY THESE

1. Suppose you invest $15000 at 10% annual interest, compounded semi-annually (*n* = 2). What will the account balance be in 10 years?

Suppose you invest $15000 at 10% interest, compounded semi-annually. You wish to know the time when your money has doubled (so the account balance is *A* = $30000).

2. Substitute all known values into the compound interest equation. There should be only one unknown, *t*. Simplify the expression.

3. Isolate the exponential by dividing each side by the principal.

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4. What is the base of the exponential?

5. Apply a logarithm of the *same base* to each side, and simplify using the fact that $\log_b(b^u)$ simplifies to $u$ (in this formula, the *base* is $b$).

6. Change the logarithm in the equation to a natural logarithm using the rule: $\log_b(x) = \frac{\ln(x)}{\ln(b)}$. Evaluate the result on your calculator.

7. Solve the equation for time, $t$, when the account balance has doubled to $30000.

## NEXT STEPS

### Continuous Growth

We have mentioned that the more times that interest is applied to a fixed-rate interest earning account, the more money you will earn. This is because more compoundings means more times for your interest to earn interest. But, is there a limit to this? What if compound interest every hour? What about every minute or second? If we could compound interest every second, our account balance would nearly be growing *continuously*!

To investigate whether our money growth is limited with increased compounding, let's look at an extraordinarily unrealistic example to see where things go.

You have found a fixed-interest earning bank account that earns interest as many times a year as you would like, at an interest rate of 100%. The only catch is that the biggest principal you can invest is $1. The

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question we need to answer is how many compoundings per year should we suggest in order to maximize your account balance.

You invest a principal of *P* = \$1. The interest rate is r = 100% = 1.00. You decide to invest the money for 1 year initially, so *t* = 1. We will experiment with different values of *n* to see how we can maximize the return on our investment. The compound interest formula becomes:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$
$$= \left(1+\frac{1}{n}\right)^{n}$$

If we decide to compound interest once, then *n* = 1, and the balance at 1 year would be

$$A = \left(1+\frac{1}{1}\right)^{1}$$
$$= 2$$

That makes sense. After 1 year at 100% interest our \$1 principal has grown to \$2. Not a very enticing investment. But, if we decide to compound more frequently, what would that do to our investment?

Let's compound interest twice, where *n* = 2. At the end of year 1 our balance is

$$A = \left(1+\frac{1}{2}\right)^{2}$$
$$= 2.25$$

Incredible! Semi-annual compounding has earned us an extra \$0.25! *Kaa-ching*! Let's look deeper. Fill in the table for account balances corresponding to increasing values of *n*.

| *n* (*compoundings*) | $A=(1+1/n)^n$ |
|---|---|
| 1 | 2 |
| 2 | 2.25 |
| 100 | |
| 10,000 | |
| 1,000,000 | |
| 10,000,000 | |

8. Is your investment growing without limit? Should you look for other ways to get rich?

9. What is the limiting value that your investment is approaching?

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## NEXT STEPS

If all went well, you learned that more compoundings per year does increase your earnings. Unfortunately, you also learned that there is a limit to those earnings. As the number of compoundings per year approaches infinity, your account balance approaches a number that mathematicians call *e*.

$$e = 2.718281828459\ldots$$

## The Base of the *Natural Exponential*

The number, *e*, is very special in mathematics. In fact, you have discovered that the mathematical expression,

$$\left(1 + \frac{1}{n}\right)^n$$

approaches *e* as *n* increases without limit.

In our example, it discovered that the balance of an account which earns interest almost continuously approaches *e*. In fact, it is not difficult to show, that if we *do* compound interest continuously, the account balance formula becomes

$$A = Pe^{rt}.$$

Here we see *e* as the base of an exponential function that models continuous growth. To compute an exponential, base *e*, on your calculator, look for the [$e^x$] key.

## TRY THESE

10. Suppose that you deposit $10000 at 7% interest into a fixed-rate continuously compounded interest account. What is the formula for the account balance?

11. What will be the account balance in 5 years?

12. Use trial and error to find the time when your account balance has doubled from the principal investment (when A = $20000).

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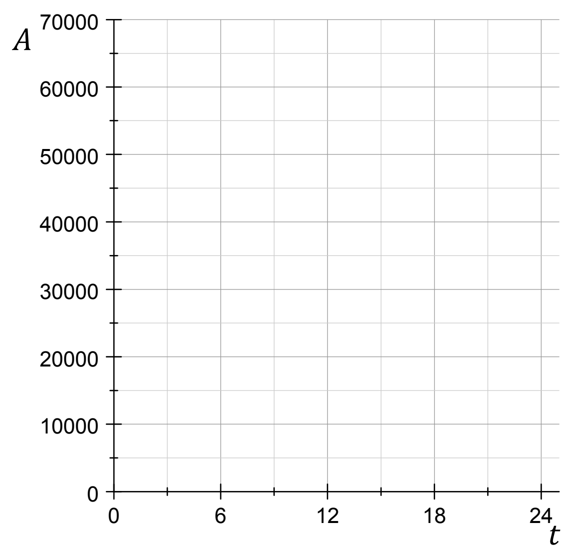
13. Compute the account balance, *A*, corresponding to the times, *t*, in the table below.

| *t* | *A* |
|---|---|
| 0 | \$10000 |
| 6 | |
| 12 | |
| 18 | |
| 24 | |

14. Plot the continuously compounded interest function below.




15. From the graph, visually estimate the time when the account balance is \$20000.

## **NEXT STEPS**

We know that every exponential function, $b^x$, there is a logarithm of the same base, $\log_b(x)$, that returns the exponent:

$$\log_b(b^u) = u.$$

Now that we have learned of the natural exponential function, $e^x$, it is time to formally define its corresponding logarithm, the natural logarithm, $\log_e(x)$. The natural logarithm is so commonly encountered that it has been given a shorthand notation: $\ln(x)$.

$$\ln(x) \text{ is shorthand for } \log_e(x).$$

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We have seen the natural logarithm several times, and have used it to evaluate other logarithms. For now, the most important fact about the natural logarithm is that it returns the exponent of the natural exponential.

$$\ln(e^u) = u.$$

**TRY THESE**

16. Refer again to our $10000 investment into a fixed 7% interest earning account. Using the continuously compounded interest formula, solve for the time when the account balance is equal to $20000. First, divide by the principal, then apply the natural logarithm to each side. Use your calculator to evaluate.

17. Does your answer match (roughly) the estimates in Problems 12 and 15?

18. When will the account balance be $30000? Use the method from Problem 16 to answer.

19. Suppose the interest rate is 10%. When will the account balance reach $100000?

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## TAKE IT HOME

There are two main formulas for calculating an account balance. The first one, shown below, is used when interest is compounded *n* times per year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

The second equation, shown below, is used when interest is compounded continuously.

$$A = Pe^{rt}$$

For the following questions, make sure you are using the correct equation. Keep an eye out for key words, when you see *continuously compounded* you know you need to use the second equation. Also, if the problem tells you that *interest is compounded quarterly or a certain number of times a year*, then you are being given a value for *n* and should be using the first equation.

1. Samuel is investing $5000 in an account that has an interest rate of 8%. How much money will Samuel have in his account after 3 years if:

   a. The interest is compounded semi-annually (2 times a year)?

   b. The interest is compounded monthly (12 times a year)?

   c. The interest is compounded weekly (52 times a year)?

   d. The interest is compounded continuously?

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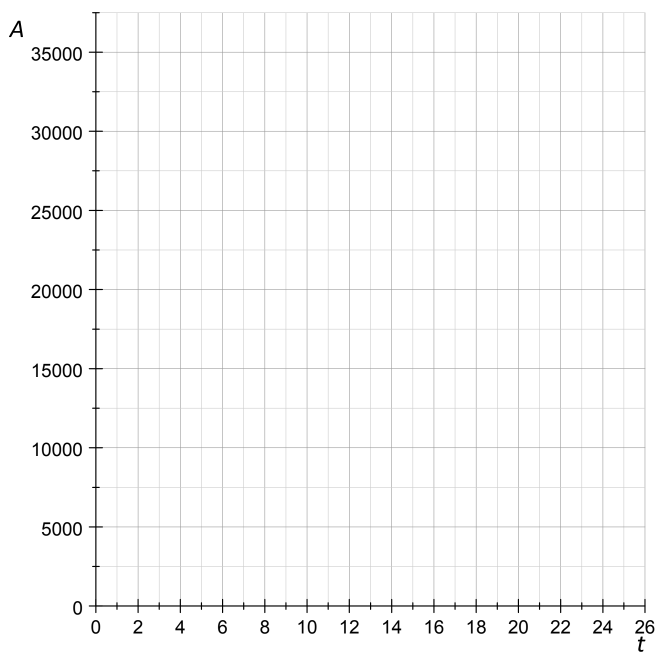
2. Bianca invests $10000 in an account that has an interest rate of 5%. The interest is compounded continuously. Use this information to answer the following questions.

   a. For each value of *t* given in the table, find the amount of money that Bianca will have in her account after that many years. Record your answers in the space provided in the table below.

| Value of *t* | *A* |
|---|---|
| 5 | |
| 10 | |
| 15 | |
| 20 | |
| 25 | |

   b. Use the values above to graph a curve that models the amount of money Bianca has in her account after *t* years.




   c. Use the graph in part (b) to approximate the number of years it will take for Bianca to have $25000 in her account.

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d. Use logarithms to determine the number of years until Bianca has $25000 in her account. (See Questions 16 – 19 in the lesson above.)

3. Thomas has $7000 to invest in an account. He has to choose between two bank accounts. Account A has a 4.5% (in decimal form that is 0.045) interest rate and the interest is compounded continuously. Account B has a 5.5% interest rate and the interest is compounded quarterly (4 times a year).

   a. How much money will be in account A if the money was invested for 2 years?

   b. How much money will be in account B if the money was invested for 2 years?

   c. Which account should Thomas invest in, if he is going to keep the money in the account for 2 years?

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4. Juan invests $12000 in an account that has a 4.25% (in decimal form that is 0.0425) interest rate. The interest is compounded continuously. How many years will it take for Juan's money to grow to $20000?

5. Samantha invested some money in an account that has a 6% interest rate. If the interest is compounded monthly, and after 5 years Samantha has $15500, how much money did she invest initially?