



Inferential Statistics and Probability a Holistic Approach

Chapter 13 Correlation and Linear Regression



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Mathematical Model

- You have a small business producing custom t-shirts.
- Without marketing, your business has revenue (sales) of \$1000 per week.
- Every dollar you spend marketing will increase revenue by 2 dollars.
- Let variable X represent amount spent on marketing and let variable Y represent revenue per week.
- Write a mathematical model that relates X to Y

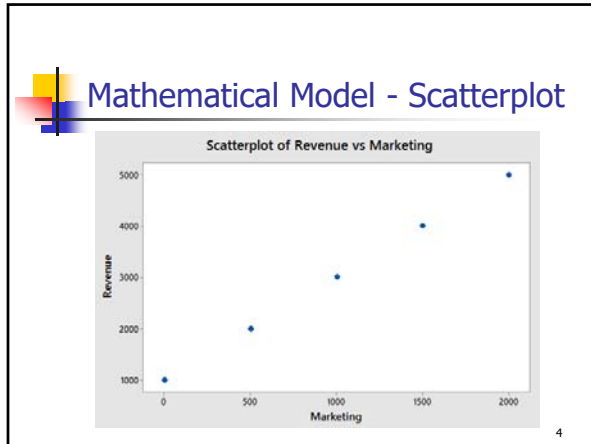
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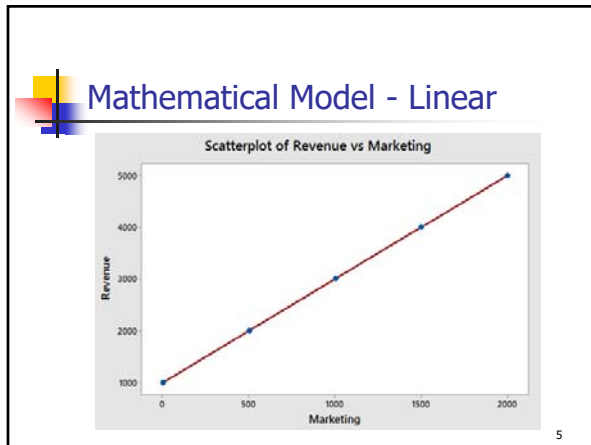


Mathematical Model - Table

X=marketing	Y=revenue
\$0	\$1000
\$500	\$2000
\$1000	\$3000
\$1500	\$4000
\$2000	\$5000

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Mathematical Linear Model

Linear Model	Example
$Y = \beta_0 + \beta_1 X$	$Y = 1000 + 2X$
Y : Dependent Variable	Y : Revenue
X : Independent Variable	X : Marketing
β_0 : Y -intercept	β_0 : \$1000
β_1 : Slope	β_1 : \$2 per \$1 marketing

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Statistical Model

- You have a small business producing custom t-shirts.
- Without marketing, your business has revenue (sales) of \$1000 per week.
- Every dollar you spend marketing will increase revenue by an expected value of 2 dollars.
- Let variable X represent amount spent on marketing and let variable Y represent revenue per week.
- Let ϵ represent the difference between Expected Revenue and Actual Revenue (Residual Error)
- Write a statistical model that relates X to Y

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Statistical Model - Table

X=Marketing	Expected Revenue	Y=Actual Revenue	ϵ =Residual Error
\$0	\$1000	\$1100	+\$100
\$500	\$2000	\$1500	-\$500
\$1000	\$3000	\$3500	+\$500
\$1500	\$4000	\$3900	-\$100
\$2000	\$5000	\$4900	-\$100

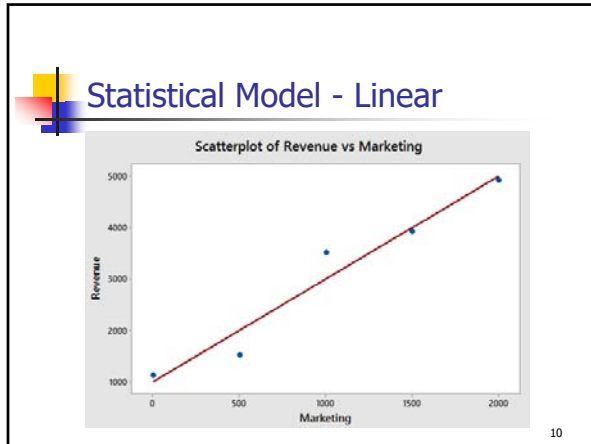
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Statistical Model - Scatterplot

Scatterplot of Revenue vs Marketing

The scatterplot shows five data points representing the relationship between marketing spend (X-axis) and actual revenue (Y-axis). The X-axis ranges from 0 to 2000, and the Y-axis ranges from 1000 to 5000. The points are approximately at (0, 1100), (500, 1500), (1000, 3500), (1500, 3900), and (2000, 4900).

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Statistical Linear Model


Regression Model	Example
$Y = \beta_0 + \beta_1 X + \varepsilon$	$Y = 1000 + 2X + \varepsilon$
Y : Dependent Variable	Y : Revenue
X : Independent Variable	X : Marketing
β_0 : Y -intercept	β_0 : \$1000
β_1 : Slope	β_1 : \$2 per \$1 marketing
ε : Normal(0, σ)	

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Regression Analysis

- Purpose:** to determine the regression equation; it is used to predict the value of the dependent variable (Y) based on the independent variable (X).
- Procedure:** select a sample from the population and list the paired data for each observation; draw a scatter diagram to give a visual portrayal of the relationship; determine the regression equation.


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Simple Linear Regression Model

$Y = \beta_0 + \beta_1 X + \varepsilon$
 Y : *Dependent Variable*
 X : *Independent Variable*
 β_0 : *Y-intercept*
 β_1 : *Slope*
 ε : *Normal (0, σ)*


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Estimation of Population Parameters

- From sample data, find statistics that will estimate the 3 population parameters
- Slope parameter
 - b_1 will be an estimator for β_1
- Y-intercept parameter
 - b_0 will be an estimator for β_0
- Standard deviation
 - s_e will be an estimator for σ

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Regression Analysis

- the regression equation: $\hat{Y} = b_0 + b_1 X$, where:
 - \hat{Y} is the average predicted value of Y for any X .
 - b_0 is the Y-intercept, or the estimated Y value when $X=0$
 - b_1 is the slope of the line, or the average change in \hat{Y} for each change of one unit in X
- the least squares principle is used to obtain b_1 and b_0

$$SSX = \sum X^2 - \frac{1}{n}(\sum X)^2 \qquad b_1 = \frac{SSXY}{SSX}$$

$$SSY = \sum Y^2 - \frac{1}{n}(\sum Y)^2$$

$$SSXY = \sum XY - \frac{1}{n}(\sum X \cdot \sum Y) \qquad b_0 = \bar{Y} - b_1 \bar{X}$$

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Assumptions Underlying Linear Regression

- For each value of X, there is a group of Y values, and these Y values are *normally distributed*.
- The *means* of these normal distributions of Y values all lie on the straight line of regression.
- The *standard deviations* of these normal distributions are equal.
- The Y values are statistically independent. This means that in the selection of a sample, the Y values chosen for a particular X value do not depend on the Y values for any other X values.

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Example


- X = Average Annual Rainfall (Inches)
- Y = Average Sale of Sunglasses/1000
 - Make a Scatterplot
 - Find the least square line

X	10	15	20	30	40
Y	40	35	25	25	15

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Example *continued*


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 **Example** *continued*

	X	Y	X ²	Y ²	XY
	10	40	100	1600	400
	15	35	225	1225	525
	20	25	400	625	500
	30	25	900	625	750
	40	15	1600	225	600
Σ	115	140	3225	4300	2775

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-  **Example** *continued*
- Find the least square line
 - SSX = 580
 - SSY = 380
 - SSXY = -445
 - $b_1 = -.767$
 - $b_0 = 45.647$
 - $\hat{Y} = 45.647 - .767X$
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 **The Standard Error of Estimate**

- The **standard error of estimate** measures the scatter, or dispersion, of the observed values around the line of regression
- The formulas that are used to compute the standard error:


$$SSR = b_1 \cdot SSXY$$

$$SSE = \sum (Y - \hat{Y})^2 = SSY - SSR$$

$$MSE = \frac{SSE}{(n - 2)}$$

$$s_e = \sqrt{MSE}$$

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Example *continued*

- Find SSE and the standard error:
 - SSR = 341.422
 - SSE = 38.578
 - MSE = 12.859
 - $s_e = 3.586$

X	Y	Y'	(Y-Yhat) ²
10	40	37.97	4.104
15	35	34.14	0.743
20	25	30.30	28.108
30	25	22.63	5.620
40	15	14.96	0.002
		Total	38.578


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Correlation Analysis

- **Correlation Analysis:** A group of statistical techniques used to measure the strength of the relationship (correlation) between two variables.
- **Scatter Diagram:** A chart that portrays the relationship between the two variables of interest.
- **Dependent Variable:** The variable that is being predicted or estimated. "Effect"
- **Independent Variable:** The variable that provides the basis for estimation. It is the predictor variable. "Cause?" (Maybe!)

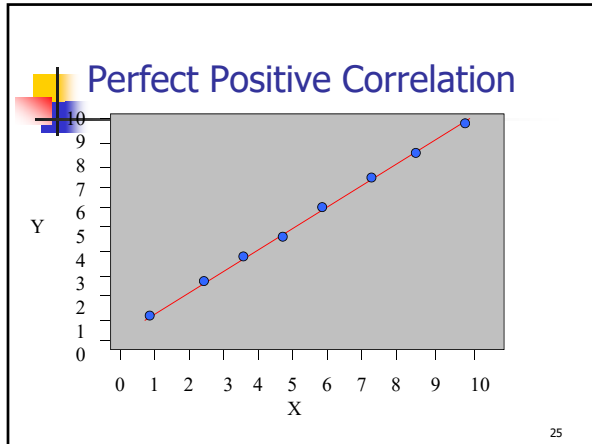
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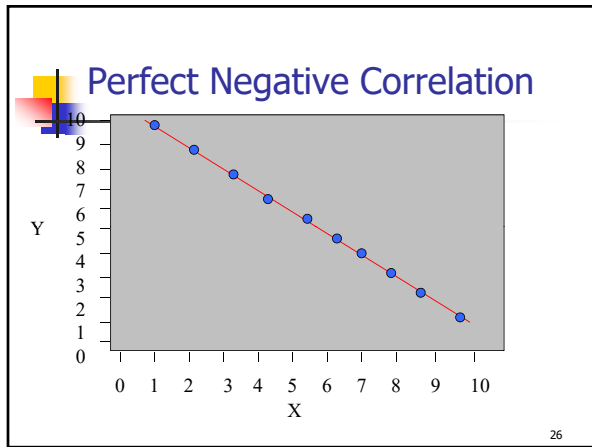


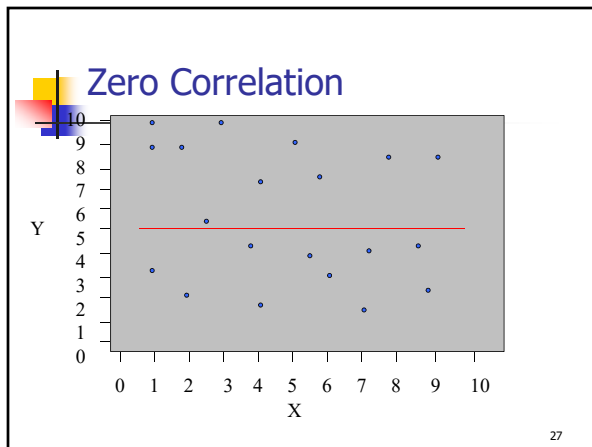
The Coefficient of Correlation, r

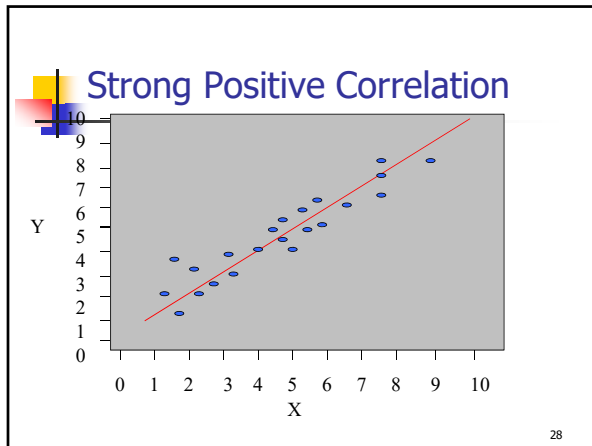
- **The Coefficient of Correlation (r)** is a measure of the **strength** of the relationship between two variables.
 - It requires interval or ratio-scaled data (variables).
 - It can range from -1.00 to 1.00.
 - Values of -1.00 or 1.00 indicate perfect and strong correlation.
 - Values close to 0.0 indicate weak correlation.
 - Negative values indicate an inverse relationship and positive values indicate a direct relationship.

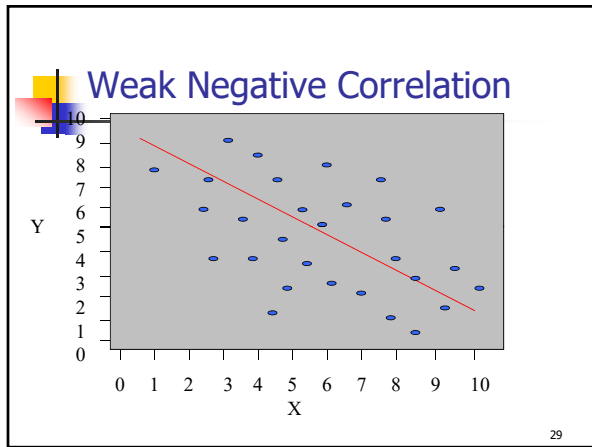
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Causation

- Correlation does not necessarily imply causation.
- There are 4 possibilities if X and Y are correlated:
 1. X causes Y
 2. Y causes X
 3. X and Y are caused by something else.
 4. Confounding - The effect of X and Y are hopelessly mixed up with other variables.


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Causation - Examples

- City with more police per capita have more crime per capita.
- As Ice cream sales go up, shark attacks go up.
- People with a cold who take a cough medicine feel better after some rest.


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r^2 : Coefficient of Determination

- r^2 is the proportion of the total variation in the dependent variable Y that is explained or accounted for by the variation in the independent variable X.
- The coefficient of determination is the square of the coefficient of correlation, and ranges from 0 to 1.

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Formulas for r and r^2

$$r = \frac{SSXY}{\sqrt{SSX \cdot SSY}} \quad r^2 = \frac{SSR}{SSY}$$

$$SSX = \sum X^2 - \frac{1}{n}(\sum X)^2$$

$$SSY = \sum Y^2 - \frac{1}{n}(\sum Y)^2$$

$$SSXY = \sum XY - \frac{1}{n}(\sum X \cdot \sum Y)$$

$$SSR = SSY - \left(\frac{SSXY^2}{SSX} \right)$$

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Example

- X = Average Annual Rainfall (Inches)
- Y = Average Sale of Sunglasses/1000

X	10	15	20	30	40
Y	40	35	25	25	15

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Example *continued*

- Make a Scatter Diagram
- Find r and r^2

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Example *continued*

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Example *continued*

X	Y	X ²	Y ²	XY
10	40	100	1600	400
15	35	225	1225	525
20	25	400	625	500
30	25	900	625	750
40	15	1600	225	600
115	140	3225	4300	2775

- $SSX = 3225 - 115^2/5 = 580$
- $SSY = 4300 - 140^2/5 = 380$
- $SSXY = 2775 - (115)(140)/5 = -445$


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Example *continued*

- $r = -445/\sqrt{580 \times 380} = -.9479$
 - Strong negative correlation
- $r^2 = .8985$
 - About 89.85% of the variability of sales is explained by rainfall.

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Characteristics of F-Distribution

- There is a "family" of F Distributions.
- Each member of the family is determined by two parameters: the numerator degrees of freedom and the denominator degrees of freedom.
- F cannot be negative, and it is a continuous distribution.
- The F distribution is positively skewed.
- Its values range from 0 to ∞ . As $F \rightarrow \infty$ the curve approaches the X-axis.

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Hypothesis Testing in Simple Linear Regression

- The following Tests are equivalent:
 - H_0 : X and Y are uncorrelated
 - H_a : X and Y are correlated
 - H_0 : $\beta_1 = 0$
 - H_a : $\beta_1 \neq 0$
- Both can be tested using ANOVA

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ANOVA Table for Simple Linear Regression

Source	SS	df	MS	F
Regression	SSR	1	SSR/dfR	MSR/MSE
Error/Residual	SSE	n-2	SSE/dfE	
TOTAL	SSY	n-1		

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Example *continued*

- Test the Hypothesis $H_0: \beta_1 = 0, \alpha=5\%$

Source	SS	df	MS	F	p-value
Regression	341.422	1	341.422	26.551	0.0142
Error	38.578	3	12.859		
TOTAL	380.000	4			

- Reject H_0 p-value < α

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Confidence Interval

- The confidence interval for the mean value of Y for a given value of X is given by:

$$\hat{Y} \pm t \cdot s_e \cdot \sqrt{\frac{1}{n} + \frac{(X - \bar{X})^2}{SSX}}$$

- Degrees of freedom for t = n-2

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Prediction Interval

- The prediction interval for an individual value of Y for a given value of X is given by:

$$\hat{Y} \pm t \cdot s_e \cdot \sqrt{1 + \frac{1}{n} + \frac{(X - \bar{X})^2}{SSX}}$$

- Degrees of freedom for t = n-2

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Example *continued*

- Find a 95% Confidence Interval for Sales of Sunglasses when rainfall = 30 inches.
- Find a 95% Prediction Interval for Sales of Sunglasses when rainfall = 30 inches.

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Example *continued*

- 95% Confidence Interval
 22.63 ± 6.60
- 95% Confidence Interval
 22.63 ± 13.18

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Using Minitab to Run Regression

- Data shown is engine size in cubic inches (X) and MPG (Y) for 20 cars.

x	y	x	y
400	15	104	25
455	14	121	26
113	24	199	21
198	22	360	10
199	18	307	10
200	21	318	11
97	27	400	9
97	26	97	27
110	25	140	28
107	24	400	15

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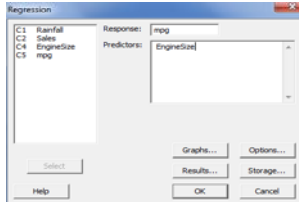
Using Minitab to Run Regression

Select Graphs>Scatterplot with regression line

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Using Minitab to Run Regression

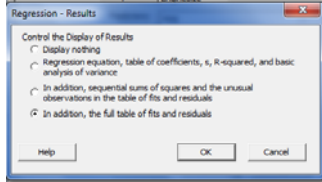
Select Statistics>Regression>Regression, then choose the Response (Y-variable) and model (X-variable)



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Using Minitab to Run Regression

Click the results box, and choose the fits and residuals to get all predictions.



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Using Minitab to Run Regression


The results at the beginning are the regression equation, the intercept and slope, the standard error of the residuals, and the r^2

The regression equation is
 $mpg = 30.2 - 0.0466 \text{ EngineSize}$

Predictor	Coef	SE Coef	T	P
Constant	30.203	1.361	22.20	0.000
EngineSize	-0.046598	0.005378	-8.66	0.000

S = 2.95688 R-Sq = 80.7% R-Sq(adj) = 79.6%

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Using Minitab to Run Regression

Next is the ANOVA table, which tests the significance of the regression model.

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	656.42	656.42	75.08	0.000
Residual Error	18	157.38	8.74		
Total	19	813.80			

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Using Minitab to Run Regression

Finally, the residuals show the potential outliers.

Obs	EngineSize	mpg	Fit	SE Fit	Residual	St Resid
1	400	15.000	11.564	1.167	3.436	1.26
2	455	14.000	9.001	1.421	4.999	1.93
3	113	24.000	24.937	0.880	-0.937	-0.33
4	198	22.000	20.976	0.673	1.024	0.36
5	199	18.000	20.930	0.672	-2.930	-1.02
6	200	21.000	20.883	0.671	0.117	0.04
7	97	27.000	25.683	0.939	1.317	0.47
8	97	26.000	25.683	0.939	0.317	0.11
9	110	25.000	25.077	0.891	-0.077	-0.03
10	107	24.000	25.217	0.902	-1.217	-0.43
11	104	25.000	25.357	0.913	-0.357	-0.13
12	121	26.000	24.565	0.853	1.435	0.51
13	199	21.000	20.930	0.672	0.070	0.02
14	360	10.000	13.427	0.998	-3.427	-1.23
15	307	10.000	15.897	0.807	-5.897	-2.07R
16	318	11.000	15.385	0.842	-4.385	-1.55
17	400	9.000	11.564	1.167	-2.564	-0.94
18	97	27.000	25.683	0.939	1.317	0.47
19	140	28.000	23.679	0.792	4.321	1.52
20	400	15.000	11.564	1.167	3.436	1.26

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